Existence of solitons in infinite lattice

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We consider necessary conditions for existence of optical solitons in one-dimensional nonlinear periodic layered array. We show analytically that in the array with the cubic-quintic nonlinearity bistable solitons are possible whereas for the Kerr nonlinearity they never exist. We investigate asymptotic behavior of the soliton amplitude at infinity. With help of the asymptotic a numerical algorithm for searching the solitons may be developed so that finding a soliton on finite interval is simultaneously the numerical proof of its existence on infinite interval.

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Localized solutions of the nonlinear Schrödinger equation with periodic potential or lattice solitons have attracted considerable attention in the last years. The objects have been considered in various fields of physics [1–5] and may be used in a variety of applications, in particular, for all-optical switching [6]. The solitons have been observed in one [7] as well as two-dimensional geometry [8], and have been predicted in higher dimensions [9]. Numerical simulations demonstrate stable propagation of the lattice solitons [10].

Recently it was revealed that single waveguide in medium with the cubic-quintic (CQ) nonlinearity can support two different solitons at the same propagation constant [11], whereas in contrast to that the Kerr (cubic) nonlinearity demonstrates one-to-one correspondence with eigenfunctions of the linear waveguide, and may be classified according to the number of zeros [12]. At first one from the solitons (low) was taken to be stable whereas others (tall) unstable. However, surprisingly, numerical simulations shown that both the solitons are stable. The study was expanded to layered waveguide array and diverse single, bistable and multistable solitons with one or few humps were found [13]. All the solitons also are stable. Moreover they can self-trap from arbitrary input.

The CQ nonlinearity was proposed, after experimental observations, as an empirical description of nonlinear dielectric response of PTS crystals [14], as well as of special chalcogenide glasses [15] and organic materials [16].

Usually the lattice solitons are considered in the framework of the nonlinear Schrödinger equation with the sinusoidal potential originated from the Gross-Pitaevskii equation [17,18]. In this paper, as a model, we consider an infinite nonlinear periodic waveguide array producing a rectangular potential (Kronig-Penney model [19]). In the model any soliton can exist only in gaps separating the Bloch bands of the linear Schrödinger equation.

Numerical calculations of spatial lattice solitons can be performed only on a finite transverse (i.e., perpendicular to the propagation direction) interval containing a finite number of potential wells. Usually the calculations are equivalent to the case of the same number of wells in homogeneous medium. Moreover the number is restricted by accuracy. For infinite lattice infinite number of one and multihumps solitons with nearby asymptotic as well as solitons with extensive gaps between humps exists and jumping from one solution to another is possible. Because of that there is no assurance that the solitons, found in the finite lattice, really exist in the infinite structure. The same is applicable to numerical simulations of stability. Therefore study of asymptotic behavior of the spatial solitons is of importance. Moreover it is well known that appropriate asymptotic allows us to simplify and decrease time of calculations since the calculations can be performed on a smaller interval.

In this paper we investigate asymptotic behavior of solitons at infinity and find a compact analytical expression for the asymptotic. We show that using the asymptotic, a numerical algorithm always can be constructed so that finding a soliton on a finite interval is simultaneously the numerical proof of its existence on infinite interval.

We define analytically necessary conditions for existence of solitons and show that bistable solitons never exist in the Kerr lattice, but are possible in lattice with the CQ nonlinearity if the cubic and quintic nonlinear constants have the opposite signs.

We start from the normalized equation for spatial solitons (see, for example, [20,21]) of the lattice with the CQ nonlinearity

$$\frac{d^2}{dx^2}R = (k - U)R - 2R^3 + R^5, \qquad (1)$$

where R is the amplitude of the electric field, k is the "propagation constant," $U(x)$ is the "potential"

$$U(x) = \begin{cases} 0 & \text{in buffer layer}, \\ U & \text{in waveguide layer}. \end{cases} \qquad (2)$$

Both $R(x)$ and its first derivative must be continuous at the buffer-waveguide interface and square integrable.

In this paper we consider single-hump solitons. However all results will be valid for multihump solitons. Indeed we may consider the multihump soliton amplitude from the last maximum to infinity and obtain the same results as for the one-hump soliton. Moreover defining the necessary conditions for bistable solitons we consider the case when only a small part of the soliton energy resides outside the waveguide layer with the soliton and the soliton amplitude is small everywhere except the layer so that we can neglect nonlinear terms in Eq. (1). The case plays the most important role in applications. Such a localization occurs near the turning points in the dependence where the number of quanta $Q \equiv 2\int_0^\infty R^2 dx$ vs the propagation constant k, whereas at brims of the dependence the soliton spreads. At the turning point $\partial Q/\partial k$ changes the sign (see Ref. [13]). In fact numerical simulations show that in the vicinity of the point the average semiwidth of the soliton $\int_0^\infty x R^2 dx / \int_0^\infty R^2 dx$ has minimal value.

Consider the asymptotic behavior of the required solutions. Place the beginning of the coordinates at the right boundary of the waveguide layer with the soliton. For this layer the period number $m=0$. In accordance with the above assumption outside the layer we may use the linear approximation. In every layer the solution of Eq. (1) without nonlinear terms satisfies

$$R = \begin{cases} C'_m \exp(ax) + C_m \exp(-ax) \text{ in buffer layer,} \\ D'_m \exp(bx) + D_m \exp(-bx) \text{ in waveguide layer,} \end{cases} \tag{3}$$

where C'_m, C_m, D'_m, D_m are constants, m is the period number, $a=\sqrt{k}$, $b=\sqrt{k-U}$, the value $m=1$ corresponds to first period on the right of the beginning of the coordinates. Required solutions do not exist if $k<0$ therefore always a is real whereas b may be as real as imaginary.

Using the boundary conditions at every interface we find in buffer layers for $m \geqslant 0$

$$\begin{aligned} C'_{m+1} &= [C'_1 H^+_{m-1} - C_1 \exp(-ad_b) B P_{m-1}] \exp(-ax_m), \\ C_{m+1} &= [C'_1 \exp(ad_b) B P_{m-1} + C_1 H^-_{m-1}] \exp(ax_m), \end{aligned} \tag{4}$$

where $x_m=(d_b+d_w)m$

$$H^\pm_{m-1} = A_\pm \exp(\pm ad_b) P_{m-1} - P_{m-2}, \tag{5}$$

$$A_\pm = \cosh(bd_w) \pm \frac{a^2+b^2}{2ab} \sinh(bd_w), \tag{6}$$

$$B = \frac{a^2-b^2}{2ab} \sinh(bd_w), \tag{7}$$

P_m is the Chebyshev polynomials: $P_m(\theta)=\sinh[(m+1)\theta]/\sinh(\theta)$, where θ is defined by $\cosh(\theta)=\xi$, $\xi \equiv \{A_+ \exp(ad_b) + A_- \exp(-ad_b)\}/2$ or using (6)

$$\xi = \cosh(ad_b)\cosh(bd_w) + \frac{a^2+b^2}{2ab} \sinh(ad_b)\sinh(bd_w). \tag{8}$$

Values $|\xi|<1$ corresponds to imaginary values of θ and define the Bloch bands of the linear Schrödinger equation where localized solutions do not exist. If $\xi>1$ then θ is real. Without loss generality we consider positive values of $\xi>1$. The transition to negative values can be achieved by change $\theta \to \theta + i\pi$. Using the definitions we obtain in buffer layers

$$\begin{aligned} R_{m+1} = {} & [C'_1 H^+_{m-1} - C_1 \exp(-ad_b) B P_{m-1}] \exp(-a\Delta_m) + \\ & [C'_1 \exp(ad_b) B P_{m-1} + C_1 H^-_{m-1}] \exp(+a\Delta_m), \end{aligned} \tag{9}$$

where $\Delta_m=(x_m-x)$. Square integrable solutions must decrease if m increases. Therefore localized solutions exist if terms with $\exp(m\theta)$ in Eq. (9) are compensated. The condition of that is

$$\begin{aligned} C'_1[A_+ \exp(ad_b) - \exp(-\theta)] - C_1 \exp(-ad_b) B &= 0, \\ C'_1 \exp(ad_b) B + C_1 [A_- \exp(-ad_b) - \exp(-\theta)] &= 0. \end{aligned} \tag{10}$$

Finally for the electric field we obtain in every buffer layer

$$R^\infty_{m+1} = C' \exp(-a\Delta_m - m\theta) + C \exp(a\Delta_m - m\theta), \tag{11}$$

where C', C is the solution of Eq. (10). By means of Eq. (11) we may easily obtain the asymptotic in the next nonlinear approximation. Integral $\int R^2 dx$ in every period decreases as $\exp(-2m\theta)$ at $m \to \infty$. For the amplitude at the waveguide-buffer and buffer-waveguide interface we obtain, respectively

$$\begin{aligned} R^{wb}_{m+1} &= (C' + C)\exp(-m\theta), \\ R^{bw}_{m+1} &= C' \exp(ad_b - m\theta) + C \exp(-ad_b - m\theta). \end{aligned} \tag{12}$$

As it follows from Eq. (10) we must also consider a special case at $B=0$. This is possible only for imaginary b. If $\xi>0$ then $b=i\pi N$, where N is an even integer, $\theta=a(d_b+d_w)$, $C'=0$, $k=U-(\pi N/d_w)^2$. (Analogously if $\xi<0$ then N is an odd integer, $C=0$.) It may be straightforwardly shown that there are no solutions having one maximum and monotonically decreasing within some waveguide layer in the Kerr lattice. However for the CQ medium this question remains open.

Let us assume now that we are looking for localized solution using, for example, the shooting method and assume that at a point x in a buffer layer the electric field and its first derivative are small enough for some initial value (for example, R_0). Then the electric field may be written as

$$\begin{aligned} R_{m+1} = {} & \delta C H^+_{m-1} [\exp(-ax_m + ax) \\ & + B P_{m-1} \exp(ad + ax_m - ax)] + R^\infty_{m+1}, \end{aligned} \tag{13}$$

where $\delta C = C'_1 - C'$ depends only from the initial value R_0 (and parameters of the structure), C' and R^∞_m is defined by Eqs. (10) and (11).

If δC changes the sign by varying the initial value then a value R_0 exists with $\delta C=0$. In this case δC equals zero at any next point by virtue of Cauchy's theorem and the solution has the required asymptotic (11). The necessary accuracy in the given linear approximation can be gained by increasing x. Usually this approximation would suffice for the solutions considered here, i.e., localized in a single waveguide layer. On the other hand the same accuracy may be obtained in the next approximations by decreasing x. The above reasoning enables us to construct a numerical algorithm so that a localized solution found on finite interval will be also the sought-for solution on infinite interval. By this means finding a solution is simultaneously the numerical proof of its existence. Varying the parameters k, U, d_w, d_b a region of the soliton existence may be defined.

Note that results of Ref. [13] found by different methods were confirmed by means of the algorithm but with the assurance that the solitons exist in infinite structure.

The necessary condition for the soliton existence is the condition for the Bloch gaps $|\xi|>1$ where solitons only can be situated. Next we state that the inequality holds

$$(R_{m+1}^{wb})^2 - (R_{m+1}^{bw})^2 > 0. \tag{14}$$

To show it denote

$$\zeta = \sinh(ad_b)\cosh(bd_w) + \frac{a^2+b^2}{2ab}\cosh(ad_b)\sinh(bd_w). \tag{15}$$

ζ is positive for real b. From equality $\xi\cosh(ad_b) - \zeta\sinh(ad_b) = \cosh(bd_w)$ it follows that if $\xi>1$ then $\zeta>0$ for imaginary b. (Analogously if $\xi<-1$ then $\zeta<0$.) Using this condition, Eq. (10) and equality

$$\zeta^2 = \xi^2 - 1 + \left(\frac{a^2-b^2}{2ab}\right)^2 \sinh^2(bd_w), \tag{16}$$

we can straightforwardly demonstrate validity of inequality (14). From this inequality and Eq. (12) it follows that the soliton amplitude decreases at every next layer-layer interface.

Consider now conditions for the maximal value of the soliton amplitude. Equation (1) is easily integrated in every waveguide and buffer layer, respectively,

$$R_{,x}^2 = \begin{cases} (k-U)R^2 - R^4 + \frac{1}{3}R^6 + \sigma_m \\ kR^2 - R^4 + \frac{1}{3}R^6 + \tau_m, \end{cases} \tag{17}$$

where σ_m, τ_m are constants. For the layer with the soliton $m=0$ and

$$\sigma_0 = -(k-U)R_0^2 + R_0^4 - \tfrac{1}{3}R_0^6, \tag{18}$$

where R_0 is the value in the soliton maximum (eigenvalue).

Using the conditions on the layer-layer interface we find for $m>0$

$$U(R_m^{wb})^2 = \sigma_{m-1} - \tau_m, \quad -U(R_m^{bw})^2 = \tau_m - \sigma_m, \tag{19}$$

Add the two expression and sum over all periods of the positive semiaxis x. Then taking into account that $\sigma_m \to 0$ if $m \to \infty$ we obtain

$$U\sum_{m=1}^{\infty}[(R_m^{wb})^2 - (R_m^{bw})^2] = \sigma_0. \tag{20}$$

$U>0$ and in accordance with Eq. (14) the difference in the brackets is positive, therefore σ_0 must be also positive. This requirement defines the lower and upper boundary for R_0

$$1.5 - \sqrt{1.5^2 - 3(k-U)} < R_0^2 < 1.5 + \sqrt{1.5^2 - 3(k-U)}. \tag{21}$$

This inequality holds if $k-U<3/4$. Initially the inequality was established analytically for the single waveguide in the CQ medium [11] and was confirmed numerically for the CQ lattice [13]. For the Kerr medium such an inequality is not valid since in this case $k-U<R_0^2$.

Consider now the condition for existence of bistable solitons. Making use of the asymptotic expression (12) we can sum all terms in the left part of Eq. (20)

$$U\sum_{m=1}^{\infty}[(R_m^{wb})^2 - (R_m^{bw})^2] = U(R_1^{wb})^2\alpha = \sigma_0, \tag{22}$$

where

$$\alpha = \frac{\{-C'^2/C^2\exp(2ad_b)+1\}[1-\exp(-2ad_b)]}{(C'/C+1)^2[1-\exp(-2\theta)]}, \tag{23}$$

C'/C is defined by Eq. (10).

Note that the right part of Eq. (17) in the waveguide layer with soliton can be written as

$$(R_0^2 - R^2)\left[-k + U + R_0^2 + R^2 - \tfrac{1}{3}(R^4 + R^2R_0^2 + R_0^4)\right]. \tag{24}$$

Let

$$R = R_0\cos\varphi, \tag{25}$$

where $0 \leq \varphi \leq \varphi_w$ and φ_w is the value at the boundary. The value with help of Eqs. (22) and (18) is defined by $U\alpha\cos^2\varphi_w = -(k-U) + R_0^2 - R_0^4/3$. Substituting (25) into Eq. (24) we can use function $\varphi(x,k,U,R_0,d_w)$ instead R

$$\varphi_{,x} = S(\varphi), \tag{26}$$

where $S(\varphi) = \sqrt{-(k-U) + R_0^2\phi_1 - R_0^4\phi_2/3}$, $\phi_1 = (1+\cos^2\varphi)$, $\phi_2 = (1+\cos^2\varphi + \cos^4\varphi)$.

Let us integrate Eq. (26)

$$\int_0^{\varphi_w}\frac{d\varphi}{S} = \frac{1}{2}d_w, \tag{27}$$

Eq. (27) defines the dependence $R_0(k,d_w,U)$. For existence of bistable and multistable solitons it is necessary that this dependence must have, as a minimum, one turning point, i.e., the point where the first derivative $\partial R_0/\partial k \to \infty$. After differentiating Eq. (27) we obtain at this point after some mathematics

$$\frac{1-\frac{2}{3}R_0^2}{U\alpha\sin(2\varphi_w)S(\varphi_w)} + \frac{1}{2}\int_0^{\varphi_w}\frac{\left(\phi_1 - \frac{2}{3}R_0^2\phi_2\right)}{S^3}d\varphi = 0. \tag{28}$$

The equality (28) holds if for an interval of φ

$$\frac{1+\cos^2\varphi}{1+\cos^2\varphi+\cos^4\varphi} < \frac{2}{3}R_0^2 < 1. \tag{29}$$

For the Kerr medium both the terms with R_0^2 in numerators of Eq. (28) must be expunged. In this case equality (28) is impossible. Therefore in the Kerr lattice bistable solitons cannot exist. The solitons also do not exist if the nonlinear constants ε_2 and ε_4 have the same sign.

The upper boundaries for the soliton amplitude (21) and (29) have impact on the process of the new hump formation. The number of humps cannot arise if the soliton has small energy. Increasing energy of the multihump soliton of the small energy leads to the energy enlargement of every soliton hump. Moreover, the number of quanta per hump, i.e., $\int_{-\infty}^{\infty} R^2 dx / N$, where N is the number of humps, practically is a constant until the turning point (see Ref. [13]). On the other hand if the soliton amplitude is near its maximal value then the new humps formation requires a large enough energy change, i.e., the formation of new soliton humps (if such a formation is possible in the given energy region) has quantum character.

In conclusion note that in addition to diverse symmetric and antisymmetric solitons found in Ref. [13] and located in waveguide layers numerical simulations show that there exist also a plethora of localized solutions with humps in buffer layers. The solutions (symmetric, antisymmetric, and asymmetric) are similar to that of single antiwaveguide in nonlinear medium. As it was found in Ref. [22] the antiwaveguide can have multivalued localized solutions. Infinite lattice may be considered as a system of waveguides ($U > 0$) or antiwavegiudes ($U < 0$) in nonlinear medium. Therefore such solutions must exist in the lattice. All the solutions apparently are unstable and fall into decay or transform into the "standard solitons" localized in the waveguide layer. Nevertheless the solutions may be used for all optical switching in a lattice where the waveguide layers consist of waveguide and antiwaveguide sections as it is described in Ref. [6].

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